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SIMILAR BOUNDARY LAYER SOLUTIONS FOR COMBINED FORCED AND FREE CONVECTION*


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# SIMILAR BOUNDARY LAYER SOLUTIONS FOR COMBINED FORCED AND FREE CONVECTION


## ABSTRACT

Consideration is given to the combined forced and free convection flow and heat transfer about a nonisothermal body subjected to a nonuniform free stream velocity. Similar solutions for the laminar boundary layer equations are found to exist when the free stream velocity and surface temperature vary respectively as $x^m$ and $x^{2m-1}$. The parameter controlling the relative importance of the free and forced convection is $Gr/Re^2$. A flow separation phenomenon may occur when the forced and free convection act in opposite directions.

Extensive numerical solutions of the transformed boundary layer equations have been carried out for the cases of uniform wall temperature and uniform wall heat flux for a Prandtl number of 0.7 (gases) over a wide range of values of $Gr/Re^2$. Results are reported for the heat transfer, shear stress, and velocity and temperature fields. Criteria are given for cataloguing flows as purely forced, purely free, and mixed.


## INTRODUCTION

The requirements of modern technology have stimulated interest in fluid flows which involve the interaction of several phenomena. Attention is directed here to the situation where forced and free convection act simultaneously in establishing the flow and temperature fields adjacent to a heated or cooled body. In approaching such a problem, a natural first step is the study of similar solutions, which constitute a set of exact solutions of the laminar boundary layer equations. Although similar solutions for the separate forced and free convection boundary layers may be found in the literature[1,2], these fundamental solutions have yet to be given for the combined flow. The finding of such solutions constitutes the goal of this investigation.

Our motivation for seeking similar solutions is three-fold. First, the results may be directly usable in important technical applications. Further, the similar solutions provide a standard of comparison against which approximate procedures may

1. V. M. Falkner and S. W. Skan, Phil. Mag., 12, 865.

2. E. M. Sparrow and J. L. Gregg, Trans. A.S.M.E., Vol. 80, 1958, 379.

be checked; once verified, the approximate methods may then be used in studying more complex flow situations where the conditions of similarity are not satisfied. Finally, the general trends may provide valuable insight in understanding the physical occurrences which take place in such combined flows.

Previous boundary layer studies [3,4] for combined forced and free convection have been confined to the isothermal vertical plate aligned parallel to a uniform free stream. These conditions do not lead to similar-type solutions. As a consequence, this problem has been studied only in an incomplete way, the published analyses taking the form of a perturbation of the pure forced convection solution. Flow in tubes and ducts under conditions of combined forced and free convection has been studied by several investigators, but their work is not of direct interest to us here and further details need not be given.

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## ANALYSIS

### General Considerations

The general problem of boundary layer flow and heat transfer will include a free stream pressure gradient and a nonisothermal surface. Previous studies have revealed similar solutions for the purely forced convection case when the free stream velocity varies according to the following power law

$$U_\infty = Ax^m \tag{1}$$

where $x$ measures the distance from the leading edge. For purely free convection flows with uniform free stream temperature $T_\infty$, similar solutions have been obtained for wall temperature variations of the type

$$T_w - T_\infty = Bx^n \tag{2}$$

Thus it would appear natural that we should begin our quest for similar solutions by simultaneously imposing power function variations for both the free stream velocity and wall temperature.

3. A. A. Tanaev, Jour. Tech. Physics (U.S.S.R.), Translated by Amer. Inst. of Physics, Vol. 1, No. 11, p. 2477, 1956.

4. E. M. Sparrow and J. L. Gregg, to be published in J. Appl. Mech.

For the sake of concreteness, we display in Figure 1 several possible physical models of the mathematical analysis. Sketches (a) and (b) show the wedge configuration commonly associated with the free stream velocity of Equation (1), while sketch (c) suggests an alternate way of achieving this velocity through the contouring of a channel wall. The same analysis covers all the situations depicted in Figure 1.

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We will designate as aiding flows those flows for which the buoyancy force has a positive component in the direction of the free stream velocity. Those flows for which the buoyancy force has a component opposite to the free stream velocity will be designated as opposing flows. For the gravity orientation shown in Figure 1, the aiding case will require $T_w - T_\infty > 0$ $(B > 0)$ and the opposed case will require $T_w - T_\infty < 0$ $(B < 0)$. If the lower half of the wedge in Figure 1(b) is considered, or if the direction of the gravity field is reversed in the other two cases, the criteria will be interchanged. Although the analysis and results are given with the orientation of Figure 1 in mind, they may be extended without essential modification to the reversed orientations.

The analysis is made for laminar, steady flow. Viscous dissipation is disregarded in consideration of the relatively small velocities usually encountered in free convection. Fluid property variations other than essential density variations are also neglected.

## The Conservation Equations

The velocity and thermal fields are governed by the basic conservation laws: mass, momentum, and energy; and it is these which constitute the starting point of our study. The boundary layer form of these equations for steady, nondissipative, constant property flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

$$\rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) = - \frac{dp}{dx} - \rho g_x + \mu \frac{\partial^2 u}{\partial y^2} \tag{4}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} \tag{5}$$

where $u$ and $v$ are the velocity components in the $x$ and $y$ directions; $T$, the static temperature; $g_x$, the magnitude of the gravity force in the negative $x$ direction; and $k$, $\rho$, $c_p$, and $\mu$, the thermal conductivity, density, specific heat, and absolute viscosity, respectively.

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The conservation of mass equation (3) may be immediately satisfied by the usual stream function $\psi$, i.e.,

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{6}$$

Then turning to equation (4), we recall that according to boundary layer theory, the pressure changes across the layer may be neglected;* and as a consequence, $dp/dx$ can be evaluated from the free stream flow. For the free stream, where the viscosity plays no role, the momentum equation is

$$\frac{dp}{dx} + \rho_\infty U_\infty \frac{dU_\infty}{dx} + \rho_\infty g_x = 0 \tag{7}$$

From this, we find an expression for $dp/dx$, which may in turn be substituted into the boundary layer momentum equation (4), giving

$$\rho\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) = \rho_\infty U_\infty \frac{dU_\infty}{dx} + g_x(\rho_\infty - \rho) + \mu \frac{\partial^2 u}{\partial y^2} \tag{4a}$$

The density difference $(\rho_\infty - \rho)$ is essential to the free convection motion and must be retained, but other density differences may be ignored within the framework of the constant property assumption. Following classical free convection theory, we rewrite the buoyancy term as

$$g_x(\rho_\infty - \rho) = g_x \beta \rho (T - T_\infty)$$

where $\beta$ is the coefficient of thermal expansion. With this equation (4a) becomes

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_\infty \frac{dU_\infty}{dx} + g_x \beta (T - T_\infty) + \nu \frac{\partial^2 u}{\partial y^2} \tag{4b}$$

where $\nu$ $(= \mu/\rho)$ represents the kinematic viscosity.

*In the present instance, the transverse pressure gradient can be neglected provided that $g_y$ is of the same or lower order than $g_x$.

A further rephrasing of the momentum equation (4b) and energy equation (5) may be made by replacing the velocity components $u$ and $v$ in favor of the stream function $\psi$ in accordance with (6). The result of this substitution is

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = U_\infty\frac{dU_\infty}{dx} + g_x\beta(T - T_\infty) + \nu\frac{\partial^3\psi}{\partial y^3} \tag{8}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} \tag{9}$$



The formidable mathematical task posed by this pair of simultaneous partial differential equations is a strong motivation to seek a similarity transformation which leads to ordinary differential equations.

The statement of the problem is completed by giving the boundary conditions. At the solid surface $(y = 0)$, the requirements that $u = v = 0$ arise respectively from the conditions of no slip and impermeability. Further, the fluid immediately adjacent to the wall is required to take on the surface temperature $T_w$. Far from the wall, the fluid velocity and temperature must asymptotically approach the free stream values $U_\infty$ and $T_\infty$. A formal statement of these boundary conditions is

$$\left.\begin{aligned} u &= \frac{\partial\psi}{\partial y} = 0 \\ v &= -\frac{\partial\psi}{\partial x} = 0 \\ T &= T_w \end{aligned}\right\} y = 0 \qquad\qquad \left.\begin{aligned} u &= \frac{\partial\psi}{\partial y} \rightarrow U_\infty \\ T &\rightarrow T_\infty \end{aligned}\right\} y \rightarrow \infty \tag{10}$$

We now proceed to reduce the partial differential equations (8) and (9) to a corresponding pair of ordinary differential equations, obtaining as permissible variations in $U_\infty$ and $T_w$ those given by equations (1) and (2).

## The Transformed Equations

Guided by previous experience with boundary layer problems, we propose a new independent variable $\eta$ which includes both the $x$ and $y$ coordinates in the following way

$$\eta = C_1 y x^\omega \tag{11a}$$

where $C_1$ and $\omega$ are constants as yet undetermined. We are further led by experience to construct new dependent variables $\theta$ and $F$ as follows

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \qquad F(\eta) = \frac{\psi}{C_2 x^\lambda} \tag{11b}$$

where again $C_2$ and $\lambda$ are unspecified. The variable $\theta$ is a dimensionless temperature and $F$ is a dimensionless stream function related to the velocities of the problem. At this point in the analysis, we cannot be certain that $\theta$ and $F$ will truly be functions of $\eta$ alone, and it is now our aim to find the conditions under which this is so.

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In a purely formal way, we may carry out the transformation of equations (8) and (9). The end result of these operations is

$$(\lambda + \omega)(F')^2 - \lambda F F'' = \frac{U_\infty \frac{dU_\infty}{dx}}{C_1^2 C_2^2 x^{2\omega+2\lambda-1}} + \frac{g_x \beta (T_w - T_\infty)\theta}{C_1^2 C_2^2 x^{2\omega+2\lambda-1}} + \nu \frac{C_1}{C_2} x^{\omega-\lambda+1} F''' \tag{12}$$

$$\left[F'\theta \frac{d \ln(T_w - T_\infty)}{d \ln x} - \lambda F \theta'\right] = \frac{k}{\rho c_p} \frac{C_1}{C_2} x^{\omega-\lambda+1} \theta'' \tag{13}$$

where primes denote differentiation with respect to $\eta$. Clearly, $F$ and $\theta$ can be functions of $\eta$ alone only if the various factors of $x$ are removed from equations (12) and (13). So, it is found from equation (13) that

$$\omega - \lambda + 1 = 0$$

and

$$\frac{d \ln (T_w - T_\infty)}{d \ln x} = \text{Constant}$$

The latter condition is a differential equation whose solution is given by Equation 2. When both conditions are inserted in Equation (12), and the criterion of independence from $x$ is applied, a differential equation for $U_\infty$ results whose solution is given by Equation (1).

It further follows that

$$n = 2m - 1 \tag{14a}$$

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and

$$\omega = \frac{m - 1}{2} = \frac{n - 1}{4}, \qquad \lambda = \frac{m + 1}{2} = \frac{n + 3}{4} \tag{14b}$$

Equation (14a) provides a very interesting result, namely, that the free stream velocity and surface temperature variations cannot be both selected arbitrarily when similar solutions are being considered. That there is a constraint tying together the variations of $T_w$ and $U_\infty$ is not surprising when we realize that the rates of boundary layer growth associated with the separate forced and free convection flows must be the same in order to achieve similar solutions for the combined flow. The boundary layer growth for pure forced convection depends on $U_\infty$, while in pure free convection the boundary layer growth depends on $T_w - T_\infty$. So, in the process of equalizing the separate boundary layer growth rates, the variations of $T_w$ and $U_\infty$ are constrained. As an example, if consideration is being given to the isothermal wall, i.e., $n = 0$, then the free stream velocity must vary as $x^{1/2}$.

The equations can now be rewritten with the $x$ dependency removed:

$$(\lambda + \omega)(F')^2 - \lambda FF'' = \frac{A^2 m}{C_1^2 C_2^2} + \frac{g_x \beta B}{C_1^2 C_2^2}\,\theta + \nu \frac{C_1}{C_2} F''' \tag{15}$$

$$nF'\theta - \lambda F\theta' = \frac{k}{\rho c_p} \frac{C_1}{C_2}\,\theta'' \tag{16}$$

Having determined the exponents $\omega$ and $\lambda$ of the transformation equations (11a) and (11b), there still remains the task of choosing the constants $C_1$ and $C_2$. Actually, there is considerable latitude in making these selections. We will explore two interesting possibilities here, the first of which will give $\eta$ and $F$ variables whose definitions coincide with pure forced convection, while the second gives an $\eta$ and $F$ whose definitions coincide with pure free convection.

First, let us select $C_2/C_1 = 2\nu$. This choice will introduce a Prandtl Number ($Pr = \mu c_p/k$) into the energy equation and reduce it to the standard form. Next, we select $F'(\infty) =$ constant, say 2. Thus we determine from Equations (1), (6), (11), and (14b), the quantities

$$C_1 = \frac{1}{2}\left(\frac{A}{\nu}\right)^{1/2}, \qquad C_2 = (A\nu)^{1/2} \tag{17a}$$

Then, using Equations (14a), (14b), and (1), we have

$$\eta = \frac{1}{2}\frac{y}{x}\left(\frac{U_\infty x}{\nu}\right)^{1/2} \tag{17b}$$

$$F = \frac{\psi}{\sqrt{U_\infty \nu x}}, \qquad u = \frac{U_\infty}{2}F', \qquad v = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\,[\eta F' - F] \tag{17c}$$

demonstrating that $\eta$ and $F$ do indeed have their forced convection definitions. Turning to Equations (15) and (16), we may then evaluate the various constant coefficients. In particular, consider the coefficient of $\theta$ in equation (15).

$$\frac{g_x \beta B}{C_1^2 C_2^2} = \text{constant}$$

$$= \frac{g_x \beta B}{A^2/4} \cdot \frac{x^{n+1}}{x^{2m}} = 4\,\frac{\left(\dfrac{g_x\beta(T_w - T_\infty)x^3}{\nu^2}\right)}{\dfrac{U_\infty^2 x^2}{\nu^2}} = 4\,\frac{Gr}{Re^2} \tag{18}$$

where we have used the definitions of Grashof and Reynolds numbers as follows

$$Gr \equiv \frac{|g_x|\,|\beta|\,|T_w - T_\infty|\,x^3}{\nu^2}, \qquad Re \equiv \frac{U_\infty x}{\nu} \tag{19}$$

and where the absolute value signs cause $Gr$ to be always positive. As is well-known, the Grashof number is the controlling parameter for the free convection flow and the Reynolds number is the controlling parameter for the forced convection flow. Not only have we been able to reduce one of the coefficients of Equation (15) to a ratio of these important parameters, but more important, we have demonstrated that $Gr/Re^2$ is independent of $x$ for similar solutions! We can then rewrite Equations (15) and (16) as

$$F''' + (m + 1)FF'' - 2m(F')^2 = 8m \pm 8(Gr/Re^2)\theta \tag{20}$$

$$\theta'' + Pr[(m + 1)F\theta' - (4m - 2)F'\theta] = 0 \tag{21}$$

where $Pr$ is the Prandtl number. The plus-minus signs in the last term of Equation (20) correspond respectively to aiding and opposing flows (the Grashof number as defined by Equation (19) is always positive).

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With the aid of equations (17c) and (11b), the boundary conditions (10) transform to

$$F(0) = F'(0) = 0, \qquad \theta(0) = 1$$
$$F' \to 2, \qquad \theta \to 0 \qquad \text{as} \qquad \eta \to \infty \tag{22}$$

So, the mathematical problem is seen to involve three parameters: $Gr/Re^2$, the exponent $m$, and $Pr$. The magnitude of $Gr/Re^2$ gives the relative importance of forced and free convection in determining the combined flow. For small values of this parameter, forced convection will predominate; while for large values, free convection will control. From Equation (20), it is clear that for opposing flows, the term $-8(Gr/Re^2)\theta$ plays the same role as does the adverse pressure gradient in the pure forced convection problem. So, it would be expected that with increasing values of $Gr/Re^2$, a situation of flow separation would reached.

As has already been remarked, the choices of $C_1$ and $C_2$ as given by Equation (17a) are by no means unique. An alternate interesting selection is obtained by setting $C_2/C_1 = 2\nu$ and the quantity $|g_x|\beta|B|/C_1^2C_2^2 = 1$. This choice leads to

$$C_1 = \left[\frac{|g_x|\beta|B|}{4\nu^2}\right]^{1/4}, \qquad C_2 = 4\left[\frac{|g_x|\beta|B|}{4\nu^2}\right]^{1/4}$$

Upon substitution into the transformation equations (11a) and (11b), it is found that $\eta$ and $F$ have definitions identical to those for a pure free convection flow. Further, the coefficients of Equations (15) and (16) may then be evaluated, giving a pair of equations for $\theta$ and $F$ which depend upon the parameters: $Re^2/Gr$, the exponent $n$ (or $m$) and $Pr$. The parameter $Re^2/Gr$ now appears in the boundary conditions as well. We will not pursue these equations here, since solutions were obtained utilizing Equations (20) and (21).

## SOLUTIONS

Inasmuch as analytical solutions of Equations (20) and (21) subject to the boundary conditions (22) could not be found, it was necessary to use numerical techniques. The numerical method, five point forward integration, requires that at the starting point of the calculation, the function and its first two derivatives be specified for a third order equation; while for a second order equation, the function and its first derivative must be prescribed. In terms of our present problem, it is necessary that $F(0)$, $F'(0)$, $F''(0)$, $\theta(0)$, and $\theta'(0)$ be specified. As is seen from the boundary conditions (22), the derivatives $F''(0)$ and $\theta'(0)$ are not known. So, the computational problem reduces to a search for the appropriate values of these derivatives which lead to solutions of Equations (20) and (21) satisfying the conditions $F' \to 2$, $\theta \to 0$ as $\eta \to \infty$. Details of the integration formulas may be found in a paper by Gregg.[5] The actual numerical work was done on an IBM 653 electronic digital computer.

Solutions of Equations (20) and (21) were carried out for a Prandtl number 0.7 (i.e., gases) for the two basic thermal boundary conditions: uniform wall temperature and uniform wall heat flux. For the former, $n = 0$ and $m = 1/2$; while for the latter, $n = 1/5$ and $m = 3/5$. The parameter $Gr/Re^2$ ranged from 0 to 100 for the aiding flow, and from 0 up to the separation value for the opposing flow.

The numerical values of $F''(0)$ and $\theta'(0)$ corresponding to these solutions have been tabulated in the Appendix. The numbers appearing there have been rounded to five significant figures. As a consequence, they do not fully represent the eight figure numbers actually used, which served to satisfy the boundary conditions for large $\eta$ to 5 or 6 decimal places.

## RESULTS

### Heat Transfer and Shear Stress

The results which may be of greatest practical interest are the heat transfer and shear stress characteristics of the problem. Considering first the heat transfer, we observe that the local rate of heat transfer $q$ (per unit area) from the surface to the fluid may be calculated using Fourier's law, i.e.,

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[5] J. L. Gregg, IBM 650 Scientific Computation Seminar, Endicott, New York, 1957.

$$q = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{23}$$

In terms of the variables of Equations (17b) and (11b), the expression for $q$ becomes

$$q = -k\,\frac{(T_w - T_\infty)Re^{1/2}}{2x}\,\theta'(0) \sim x^{(5n-1)/4} \tag{23a}$$

where we have used the definition of Reynolds number from (19). It is customary to rephrase the heat transfer results in terms of heat transfer coefficients and Nusselt numbers according to the following definitions

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$$h \equiv \frac{q}{T_w - T_\infty}, \qquad Nu \equiv \frac{hx}{k} \tag{24}$$

With this, we arrive at the dimensionless heat transfer representation

$$Nu/Re^{1/2} = -\left(\frac{1}{2}\right)\theta'(0) \tag{25}$$

Since $\theta'(0)$ depends upon $Gr/Re^2$, m and Pr, so then does the Nusselt-Reynolds relationship.

Passing now to the wall shear stress $\tau_w$, we note that

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{26}$$

Defining a friction coefficient $c_f$ by the relation

$$c_f = \frac{\tau_w}{(\rho U_\infty^2/2)} \tag{27}$$

and introducing the variables of the analysis, we find the dimensionless representation of the shear stress results to be

$$c_f Re^{1/2} = \left(\frac{1}{2}\right)F''(0) \tag{28}$$

The relation between $c_f$ and Reynolds number depends upon $Gr/Re^2$, m, and Pr.

Turning to the numerical results, it may be observed that the values of $\theta'(0)$ and $F''(0)$ tabulated in the Appendix are directly usable in the dimensionless heat transfer and friction expressions. Our aim is to illustrate general characteristics by means of plots. Inasmuch as the general trends for the uniform wall temperature case appear to be repeated for the uniform heat flux case, there appears to be no need for detailed consideration of both. So, we will focus primary attention on the results for uniform wall temperature.

On figure 2, we plot the dimensionless heat transfer and shear stress parameters of Equations (25) and (28) as a function of $Gr^{1/2}/Re$ for the situation of aiding flows. For small values of the abscissa, the flow is close to pure forced convection; while for large values of the abscissa, the flow is approaching free convection. With this in mind, there are included lines representing the results for the pure forced convection and the pure free convection flows. The equations of these limiting lines are:

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$$\left.\begin{aligned} 4Nu/Re^{1/2} &= 1.665 \\ 2c_f Re^{1/2} &= 3.599 \end{aligned}\right\} \text{Forced convection} \tag{29a}$$

$$\left.\begin{aligned} 4Nu/Re^{1/2} &= 1.413(Gr^{1/2}/Re)^{1/2} \\ 2c_f Re^{1/2} &= 3.840(Gr^{1/2}/Re)^{3/2} \end{aligned}\right\} \text{Free convection}^{*} \tag{29b}$$

The interesting feature of this plot is that the heat transfer results for the mixed flow show a surprisingly small deviation from the envelope formed by the two limiting lines. In fact, the heat transfer predictions based on the envelope lines would be in error at most by 23 percent. This somewhat unexpected finding has important practical implications. For the skin friction results, the maximum deviations from the envelope curve are much larger. Based on the curves of Figure 2, we will later give criteria to distinguish when a flow may be considered as purely forced, purely free, or mixed.

The phrasing of the results as Nusselt-Reynolds and friction factor-Reynolds relations may imply that the problem is being viewed as a basic forced convection flow on which there is superposed varying amounts of free convection. By recasting the results in terms of free convection correlation variables, the opposite viewpoint may be obtained. Such a step has been taken on Figure 3, where we plot $Nu/Gr^{1/4}$ and

*It may be easily verified that $U_\infty$ drops out of the expression for $\tau$.

greater free convection effect is required to bring about separation for uniform heat flux may be explained by noting that the free stream flow exerts a stronger positive pressure gradient in this case, i.e., $U_\infty \propto x^{0.6}$ as opposed to $U_\infty \propto x^{0.5}$. The heat transfer and friction factor results for the uniform heat flux (UHF) and uniform wall temperature (UWT) cases are compared on Figure 5. For aiding flows, the Nusselt numbers for the two situations are in an almost constant ratio (1.175-1.141) over the entire range of $Gr^{1/2}/Re$. For opposing flows, the variation of the Nusselt number ratio is somewhat greater. The friction factor ratio ranges from 1.083 to 0.94 for aiding flows and, as expected, displays a significantly greater variation for opposing flows.

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## Velocity and Temperature Profiles

Greater insight into the details of the flow and thermal fields may be obtained from the velocity and temperature profiles. Again, we deal with only the uniform wall temperature situation, noting that the same trends appear for uniform heat flux.

The distribution of the velocity across the boundary layer has been plotted on Figure 6 for a wide range of the parameter $Gr/Re^2$. This series of curves strikingly displays the role of the free convection in modifying the flow field. As $Gr/Re^2$ ranges from 0 to 100 under aiding flow conditions, the profiles change from the pure forced convection to a shape which is, aside from the tail of the curve, essentially that of free convection. For opposing flows, as $Gr/Re^2$ ranges from 0 to 0.95, the increasingly significant undermining effects of the free convection are clearly visible. The shapes of these latter curves are quite similar to those for pure forced convection flow under an adverse pressure gradient.

Boundary layer temperature distributions are given on Figure 7. All curves display the same simple shape which is also found in the thermal boundary layers of the pure forced and pure free convection flows.

## Criteria for Pure and Mixed Flows

It is of practical interest in the computation of heat transfer and shear stress to distinguish those conditions under which a given flow may be regarded as pure

$\dfrac{\tau_w}{\rho\left(\frac{\nu}{x}\right)^2} \Big/ Gr^{3/4}$ as a function of $Re/Gr^{1/2}$. On this graph, small values of the abscissa correspond to predominantly free convection flows, while large values of the abscissa correspond to nearly forced convection flows. The limiting lines corresponding to pure forced and free convection are also shown, and their equations are given as follows

$$\left.\begin{aligned} 4Nu/Gr^{1/4} &= 1.413 \\ \left[4\tau_w \Big/ \rho\left(\frac{\nu}{x}\right)^2\right] Gr^{-3/4} &= 3.840 \end{aligned}\right\} \text{Free convection} \qquad (30a)$$

$$\left.\begin{aligned} 4Nu/Gr^{1/4} &= 1.665(Re/Gr^{1/2})^{1/2} \\ \left[4\tau_w \Big/ \rho\left(\frac{\nu}{x}\right)^2\right] Gr^{-3/4} &= 3.599(Re/Gr^{1/2})^{3/4} \end{aligned}\right\} \text{Forced convection} \qquad (30b)$$

The general remarks about Figure 2 apply as well to Figure 3.

So far we have been concerned with aiding flows. Now, we turn to opposing flows. As has been already noted, the increasing free convection effects associated with increasing $Gr/Re^2$ should lead to separation. So, it is expected that $c_f$ should decrease monotonically to zero as $Gr/Re^2$ increases. The dimensionless heat transfer and friction factor results for the opposing flow situation are plotted in Figure 4 for the isothermal wall case. The friction factor decreases as we have supposed, the separation point occurring at a value of $Gr/Re^2$ very slightly in excess of 0.95. But, the interesting point is the relatively minor variation experienced by the dimensionless heat transfer parameter, showing a decline of only 28 percent over the whole range from pure forced convection ($Gr/Re^2 = 0$) to separation. Again, this finding may have important practical implications.

Plots depicting the results for the uniform heat flux situation would be essentially the same as Figures 2, 3, and 4, the only differences being in the numerical values. We now proceed to summarize the numerical differences. First of all, the separation point for the uniform heat flux case occurs at $Gr/Re^2 = 1.15$, while the separation value for uniform wall temperature is 0.95. The fact that a relatively (either forced or free) from those under which it must be regarded as mixed. The importance of this matter stems from the fact that frequently, only results for the pure flows are available.

First we examine the conditions under which the free convection effects may be neglected in computing the local heat transfer. Our goal is to find a quantitative criterion for determining when the pure forced convection heat transfer relationship will yield sufficiently accurate results for the mixed flow case. Suppose it is decided that an accuracy of 5 percent is adequate for the great majority of applications. Then, using Figure 2, we find that when $Gr/Re^2 < 0.3$, the flow can be regarded as purely forced as far as the heat transfer computation is concerned.

Next, we inquire about the conditions under which the free convection heat transfer relationship will yield sufficiently accurate results for the mixed flow situation. Again supposing that an accuracy of 5 percent is adequate, and using Figure 3, it is found that the pure free convection results can be used when $Gr/Re^2 > 16$.

So, in summary, the following subdivisions can be made from the standpoint of the heat transfer computation.

$$0 < Gr/Re^2 < 0.3 \qquad \text{Forced convection} \qquad (31a)$$

$$0.3 < Gr/Re^2 < 16 \qquad \text{Mixed flow} \qquad (31b)$$

$$16 < Gr/Re^2 \qquad \text{Free convection} \qquad (31c)$$

These criteria were derived for the case of aiding flows. For opposing flows, we find that Equation (31a) still applies, but we must now replace (31b) by the relation

$$0.3 < Gr/Re^2 \qquad \text{Mixed flow} \qquad (31d)$$

and note that no criterion for purely free convection flow is available since solutions were not obtained beyond the separation point.

Just as Equation (31) catalogues the flow type on the basis of heat transfer computations, so too may the flow be catalogued on the basis of shear stress computations. Again adopting a figure of 5 percent as representing an acceptable accuracy, the flow subdivisions from the standpoint of the shear stress are found to be

[redacted]



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## APPENDIX

### TABULATION OF F"(0) AND θ'(0) VALUES

(a) Aiding flows.

| $Gr/Re^2$ | UWT | | UHF | |
|---|---|---|---|---|
| | F"(0) | -θ'(0) | F"(0) | -θ'(0) |
| 100 | 122.59 | 2.2698 | 116.02 | 2.5882 |
| 50 | 73.449 | 1.9251 | 69.589 | 2.1952 |
| 20 | 37.733 | 1.5613 | 35.871 | 1.7816 |
| 10 | 23.187 | 1.3466 | 22.165 | 1.5389 |
| 5 | 14.652 | 1.1784 | 14.145 | 1.3504 |
| 25/9 | 10.290 | 1.0690 | | |
| 2 | | | 8.5063 | 1.1761 |
| 1 | 6.2804 | .94252 | 6.3457 | 1.0927 |
| .8 | 5.7798 | .92401 | 5.8853 | 1.0732 |
| .5 | 5.0001 | .89353 | 5.1712 | 1.0412 |
| .25 | 4.3183 | .86492 | 4.5507 | 1.0117 |
| .05 | 3.7463 | .83927 | 4.0338 | .98574 |
| .02 | 3.6581 | .83516 | 3.9545 | .98162 |
| 0 | 3.5989 | .83238 | 3.9013 | .97883 |

(b) Opposing flows.

| $Gr/Re^2$ | UWT F"(0) | UWT -θ'(0) | UHF F"(0) | UHF -θ'(0) |
|---|---|---|---|---|
| 0 | 3.5989 | 0.83238 | 3.9013 | 0.97883 |
| .02 | 3.5394 | .82956 | 3.8479 | .97602 |
| .05 | 3.4495 | .82527 | 3.7673 | .97175 |
| .1 | 3.2980 | .81792 | 3.6318 | .96447 |
| .25 | | | 3.2152 | .94134 |
| .5 | 1.9852 | .74722 | 2.4795 | .89728 |
| .8 | .79060 | .66676 | 1.4957 | .83003 |
| .9 | .29531 | .62582 | | |
| .94 | .065485 | .60449 | | |
| .95 | .0033788 | .59841 | | |
| 1.0 | | | .73003 | .76816 |
| 1.1 | | | .28082 | .72610 |
| 1.15 | | | .024373 | .69943 |
| 1.1544 | | | .00025963 | .69680 |

$$0 < Gr/Re^2 < 0.06 \qquad \text{Forced convection} \qquad (32a)$$

$$0.06 < Gr/Re^2 < 16 \qquad \text{Mixed flow} \qquad (32b)$$

$$16 < Gr/Re^2 \qquad \text{Free convection} \qquad (32c)$$

These criteria are based on the results for aiding flows. Equation (32a) also applies to opposing flows, but (32b) is replaced by

$$0.06 < Gr/Re^2 \qquad \text{Mixed flow} \qquad (32d)$$

As before, no criterion has been obtained for purely free convection flow under opposing conditions.

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The criteria embodied in Equations (31) and (32) were derived using the results of the uniform wall temperature case, but they also apply rather well to the uniform heat flux situation.

## OTHER SIMILARITY SITUATIONS

Consideration was given to the situation where the free stream velocity of Equations (1) takes on negative values, i.e., $A < 0$. Physically speaking, this represents the unusual case of a free stream flow toward the leading edge. For this situation, the conservation equations can be reduced identically to the ordinary differential equations (20) and (21), provided only that minus signs be inserted in the brackets of Equation (17a) determining $C_1$ and $C_2$. The boundary conditions (22) still apply, except that now $F' \to -2$ as $\eta \to \infty$. In studying the signs of the various terms of Equations (20) and (21) for large values of $\eta$, it is found that contradictions occur. We are thus persuaded that physically reasonable solutions cannot be found for the negative free stream situation.

It has also be demonstrated that the conservation equations reduce to ordinary differential equations when $U_\infty \propto e^{ax}$, $T_w - T_\infty \propto e^{bx}$. Detailed study and numerical computations, however, have not been carried out.

## ACKNOWLEDGMENT

The authors acknowledge the encouragement and numerous helpful suggestions of Dr. E. R. G. Eckert.

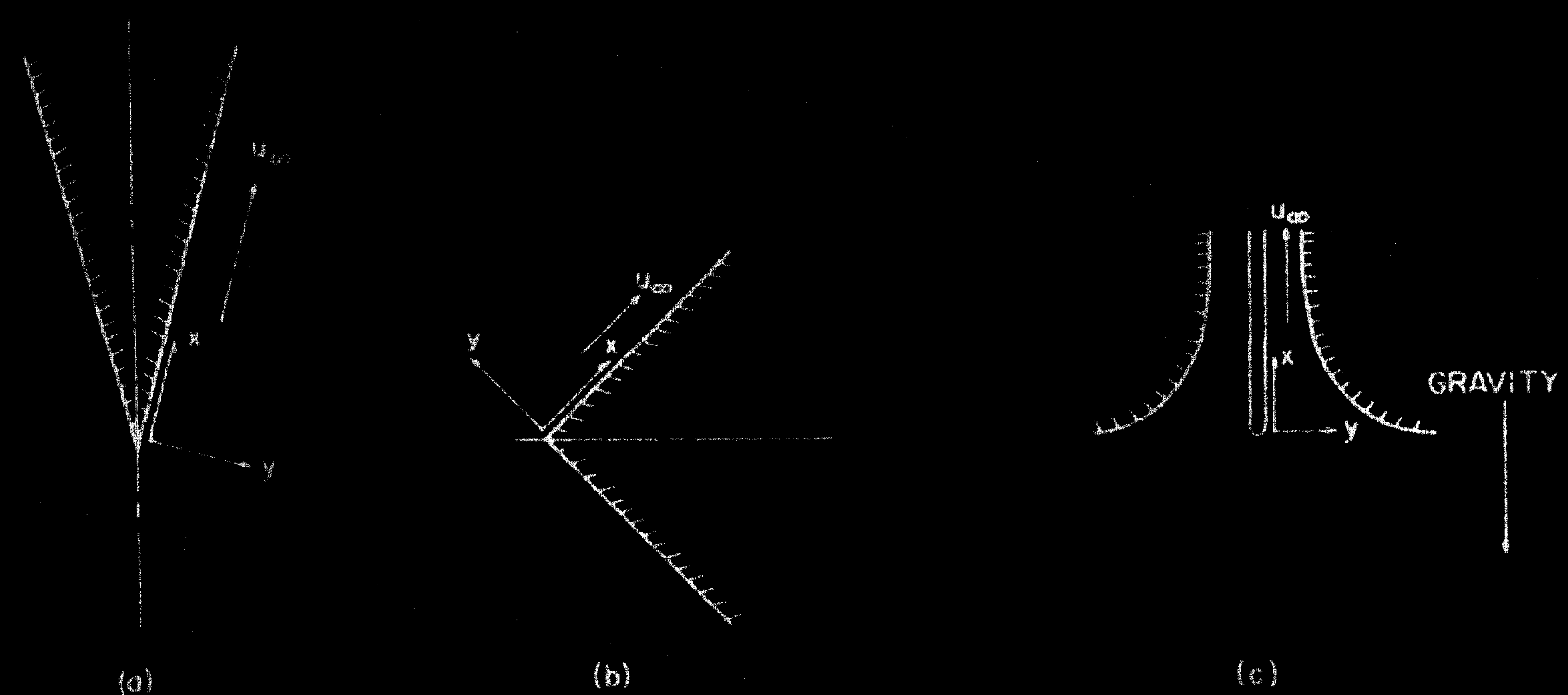


FIGURE 1

List of Figures

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Fig. 4. - Heat transfer and friction factor results for opposing flows (uniform wall temperature, Pr = 0.7).

Fig. 5. - Comparison of uniform wall temperature (UWT) and uniform heat flux (UHF) results.

Fig. 6. - Representative velocity profiles (uniform wall temperature, Pr = 0.7).

Fig. 7. - Representative temperature profiles (uniform wall temperature, Pr = 0.7).

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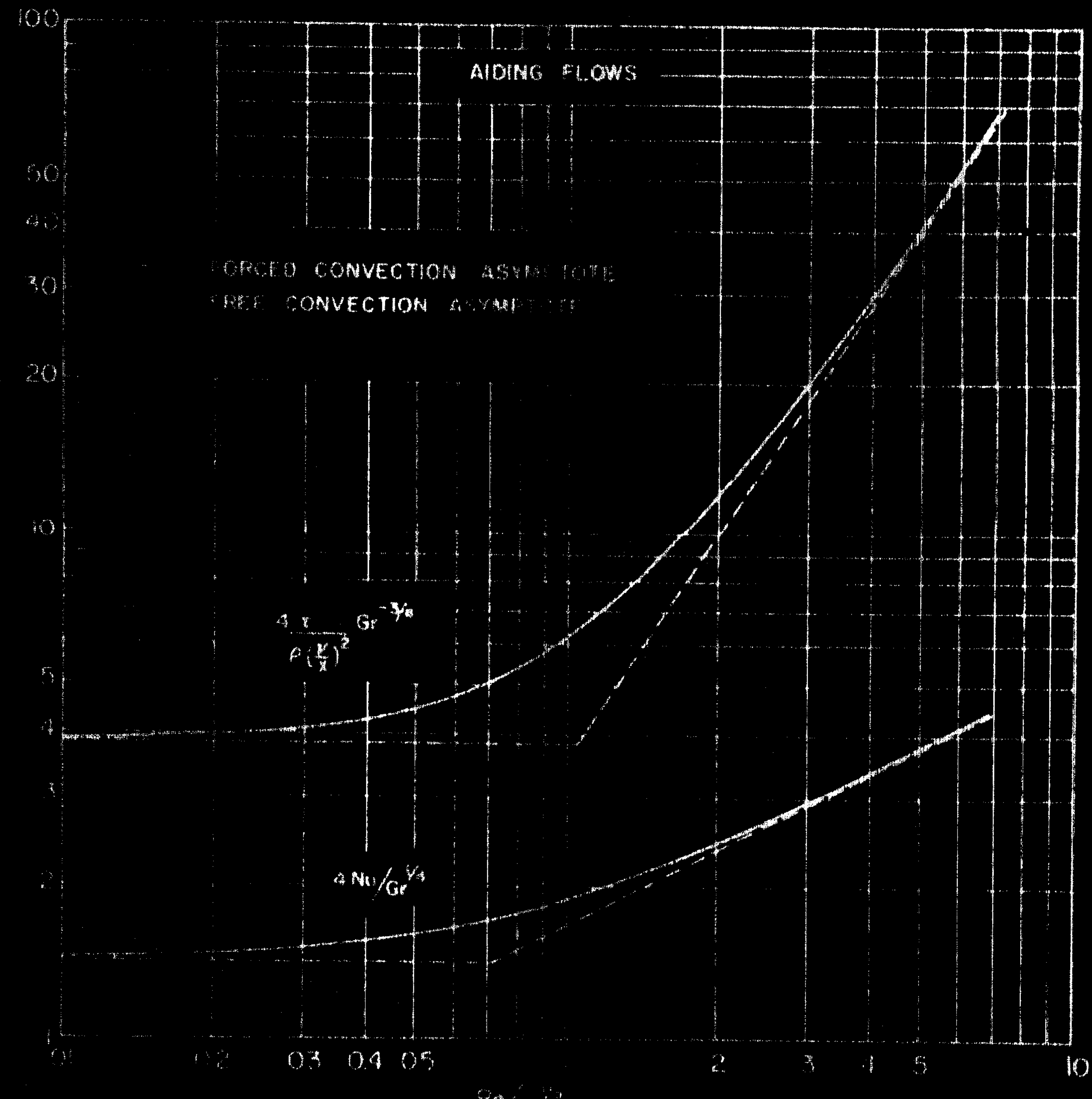
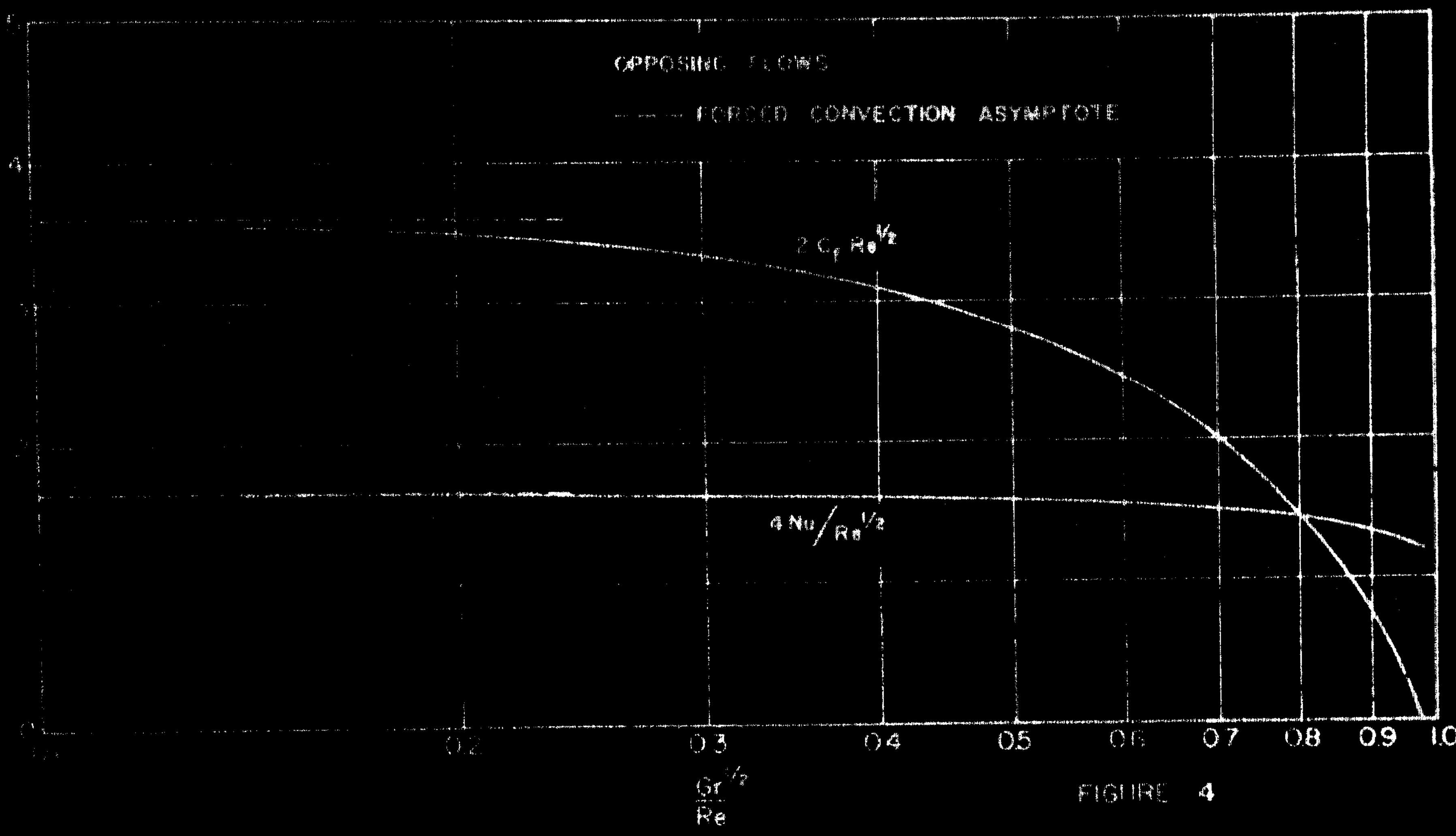

AIDING FLOWS
FORCED CONVECTION ASYMPTOTE
FREE CONVECTION ASYMPTOTE
$\frac{4\tau}{\rho\left(\frac{\nu}{x}\right)^2} Gr^{-3/4}$
$4\,Nu/Gr^{1/4}$
100
50
40
30
20
10
5
4
3
2
1
0.1
0.2
0.3
0.4
0.5
2
3
4
5
10
$Re/Gr^{1/2}$

FIGURE 5



OPPOSING FLOWS
FORCED CONVECTION ASYMPTOTE
$2\,C_f\,Re^{1/2}$
$4\,Nu/Re^{1/2}$
4
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\frac{Gr^{1/2}}{Re}$


FIGURE 4

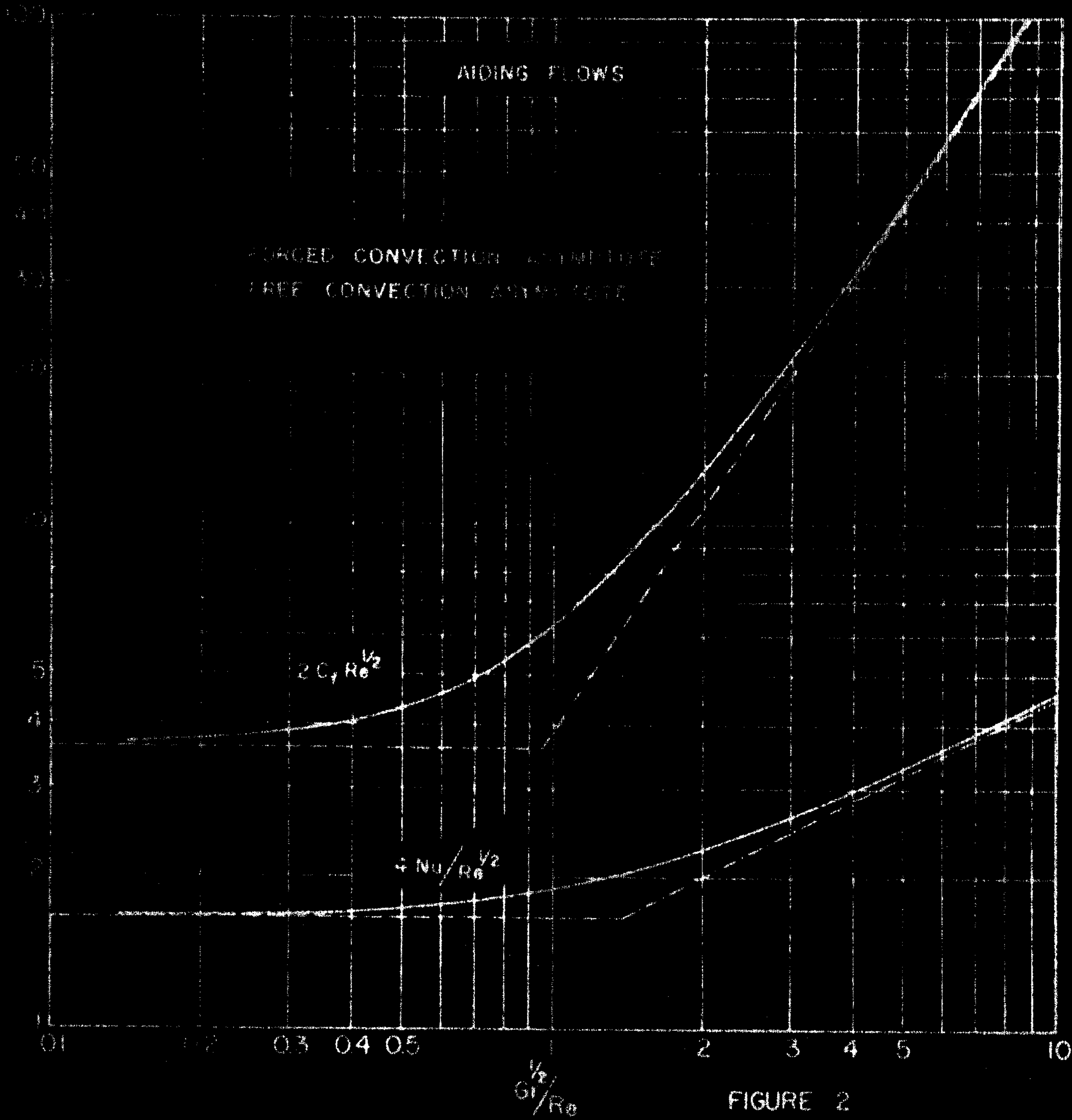

AIDING FLOWS
CONVECTION
CONVECTION
2 Cf Re1/2
4 Nu/Re1/2
5
4
3
2
1
.01
0.3 0.4 0.5
2
3
4
5
10
Gr1/2/Re

FIGURE 2

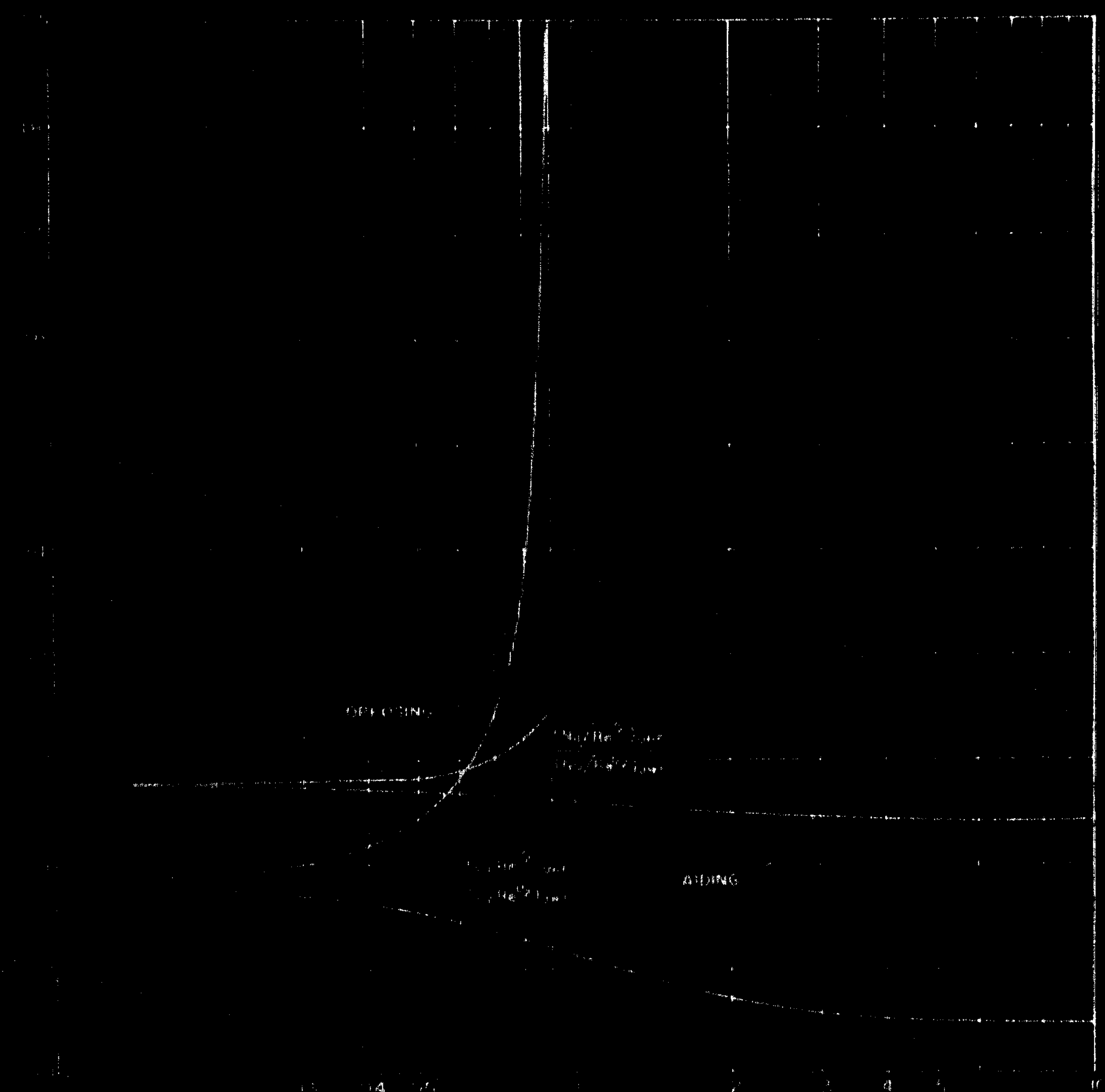



FIGURE 5

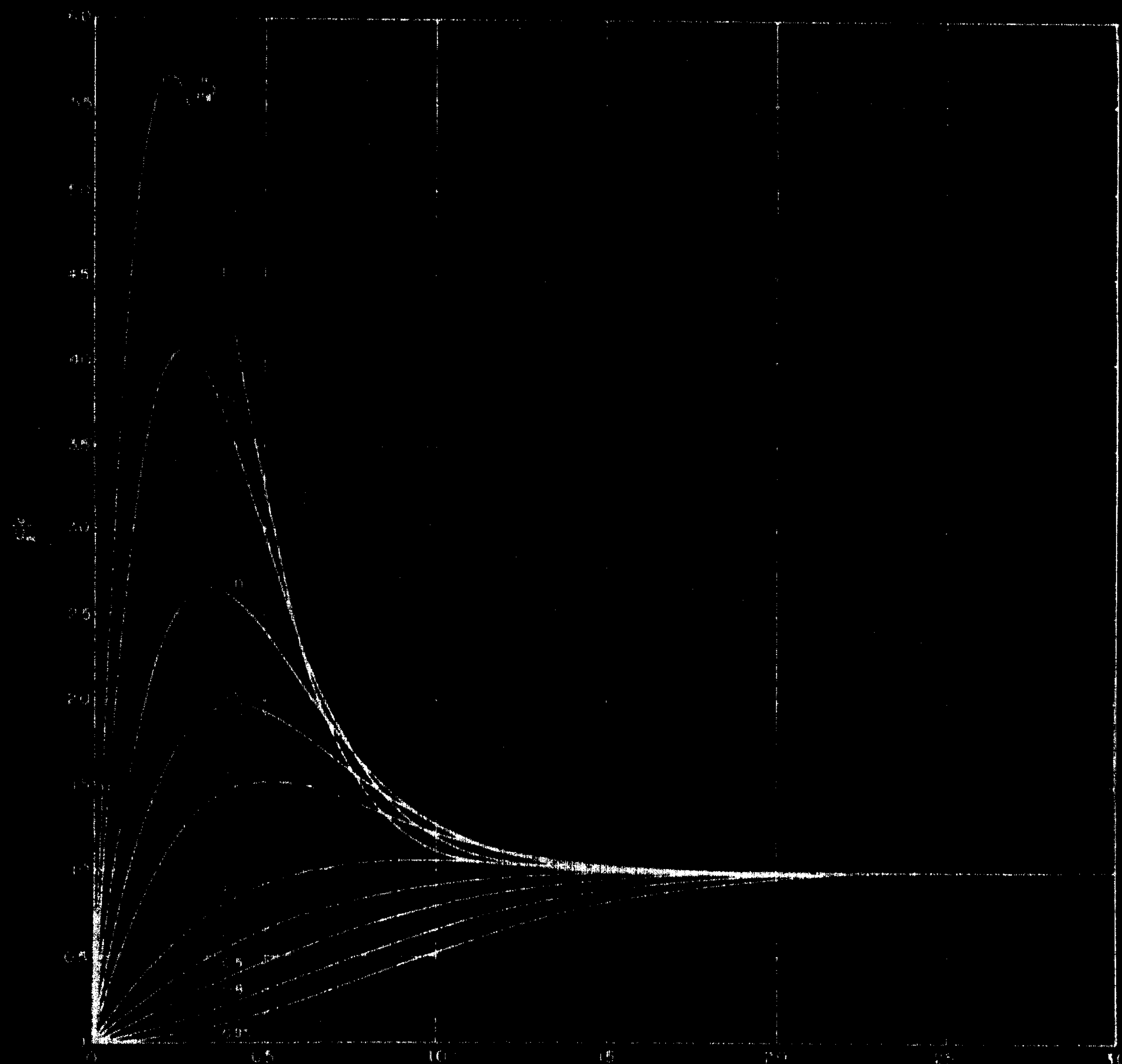



FIGURE 6